Rotation Averaging and Strong Duality

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Structure from Motion

Visual Navigation

Visual Localization
Main topic: Semidefinite relaxations for optimization over $SO(3)$

- Introduction

- Problem formulation and examples

- Analysis: Relaxations, tightness and extreme points

- In depth: Rotation averaging

- Conclusions
- Goal: Recover camera poses given relative pairwise measurements

\[
\arg \min_{R_1, \ldots, R_n \in SO(3)} \sum_{(i,j) \in E} \| R_i \tilde{R}_{ij} - R_j \|_F^2
\]
Hand-eye calibration

\[ \min_{R \in SO(3)} \sum_{i=1}^{m} \left\| A_i R - RB_i \right\|_F^2 \]
The Chordal distance

- Defined as the Euclidean distance in the embedding space,

\[ d(R, S) = \| R - S \|_F \]

- Equivalent to:

\[ d(R, S) = 2\sqrt{2} \sin \frac{|\alpha|}{2} \]
Registration of points, lines and planes

\[
\min_{R \in \text{SO}(3), t} \sum_{i=1}^{m} \| P_i (R x_i + t - y_i) \|^2
\]
Problem formulation

Let

\[ R = [R_1, \ldots, R_n], \]

where each \( R_i \in SO(3) \).

\[ \min_{R \in SO(3)^n} \begin{bmatrix} \text{vec}(R) \\ 1 \end{bmatrix}^T Q \begin{bmatrix} \text{vec}(R) \\ 1 \end{bmatrix} \]
How to overcome the problem of non-convexity?

- One idea: Relax some constraints and solve relaxed problem

- How to relax?
  1. Linearize
  2. Convexify

- Tightness: When is the solution to the original and relaxed problem the same?
- Longuet-Higgins, 1981
- Stefanovic, 1973
- Thompson, 1959
- Chasles, 1855
- Hesse, 1863
- Hauck, 1883
Convexification

- Quasi-convexity
  
  Q. Ke, T. Kanade, PAMI 2007
  F. Kahl, R. Hartley, PAMI 2008

- Semidefinite relaxations
  
  F. Kahl, D. Henrion, IJCV 2007
  C. Aholt, S. Agarwal, R. Thomas, ECCV 2012
Estimating a single rotation

\[
\min_{\Lambda} \quad [r \ 1]^T Q [r \ 1]
\]

\[
r = \text{vec}(R), \; R \in SO(3)
\]

Set \( \Lambda = [r \ 1][r \ 1]^T \)

\[
\min_{\Lambda} \quad \text{tr}(Q \Lambda)
\]

\[
\Lambda \succeq 0, \; \text{tr}(A_i \Lambda) = b_i
\]

Convex program, but ignores \( \text{rank}(\Lambda) = 1! \)
Estimating a single rotation

Original problem

\[ \min_{R} \quad [r_1]^T Q [r_1] \]
\[ r = \text{vec}(R), \quad R \in SO(3) \]

Relaxed problem

\[ \min_{\Lambda} \quad \text{tr}(Q \Lambda) \]
\[ \Lambda \succeq 0, \quad \text{tr}(A_i \Lambda) = b_i \]

- Is the relaxation always tight?

- Are all minimizers \( \Lambda^* \) of the convex relaxation rank one?
Empirical result for 1000 random Q:s
Algebraic Geometry to the rescue
Multi-variate polynomial $p(r)$ is a sums of squares (SOS) if

$$p(r) = \sum_i p_i^2(r)$$

Let $R^* \in X$ be a minimizer with optimal value $q^*$. Is $\begin{bmatrix} r^T \end{bmatrix} Q \begin{bmatrix} r \\ 1 \end{bmatrix} - q^*$ a sum of squares?
Let $X$ be the variety of $3 \times 3$ matrices of $SO(3)$.

$$SO(3) \in \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \ \det(R) = 1 \}$$

$$\dim(X) = 3, \ \text{codim}(X) = 6, \ \text{degree}(X) = 8.$$ 

**Definition:** If non-degenerate variety $X$ has $\text{codim}(X) + 1 = \text{degree}(X)$ then it is called *minimal*. 
Theorem. Every real quadratic form that is non-negative on the non-degenerate variety $X$ is a sum of squares of linear forms if and only if $X$ is a variety of minimal degree.

$$\dim(X) = 3, \text{ codim}(X) = 6, \text{ degree}(X) = 8.$$  

But minimal degree $= \text{ codim} + 1 = 7 < 8$. 
Extreme points

Original problem

\[
\min_R \begin{bmatrix} r \\ 1 \end{bmatrix}^T Q \begin{bmatrix} r \\ 1 \end{bmatrix}
\]

\[
r = \text{vec}(R), \ R \in SO(3)
\]

Relaxed problem

\[
\min_\Lambda \quad \text{tr}(QA) \quad \Lambda \succeq 0, \ \text{tr}(A_i \Lambda) = b_i
\]

- Are all minimizers \( \Lambda^* \) of the convex relaxation rank one?

**Theorem:** Every extreme point \( \Lambda^* \) has \( \text{rank}(\Lambda^*) = 1 \) or \( \text{rank}(\Lambda^*) = 6 \).
Empirical result for 1000 random $Q$:s for $SO(3) \times SO(3)$
Rotation averaging in Structure from Motion
A possible pipeline:

1. Estimate relative epipolar geometries (5-point algorithm)
2. Given relative rotations, estimate absolute rotations
3. Compute camera positions and 3D points ($L_\infty$-optimization)
- Goal: Recover camera poses given relative pairwise measurements
- Quaternions:

  V.M. Govindu, *CVPR 2001*

- Single rotation estimation:


- Duality:

  A. Singer, *Applied and Computational Harmonic Analysis, 2011*

  J. Fredriksson, C. Olsson, *ACCV 2012*


- Analysis:

  K. Wilson, D. Bindel and N. Snavely, *ECCV 2016*
- Problem formulation

\[
\arg \min_{R_1, \ldots, R_n \in SO(3)} \sum_{(i,j) \in E} \| R_i \tilde{R}_{ij} - R_j \|^2_F
\]

Graph \((V, E)\) where \(V\) = camera poses and \(E\) = relative rotations
- Problem formulation

$$\arg \min_{R_1, \ldots, R_n \in SO(3)} \sum_{(i,j) \in E} \left\| R_i \tilde{R}_{ij} - R_j \right\|_F^2$$

Graph $\Gamma(V, E)$ where $V =$ camera poses and $E =$ relative rotations
- Non-convex problem

Ground truth

Local minimum

Rotation averaging

Objective function

?
- Background
  - Well established theory on duality for convex optimization
  - Duality is at the core of many existing optimization algorithms
  - Less understood about the non-convex case
- Aims
  - Can we obtain guarantees of global optimality?
  - How to design efficient optimization algorithms?
Recall

\[
\min_x f(x) \quad \geq \quad \min_x f(x) + \lambda h(x)
\]

s.t. \quad h(x) = 0

What \( \lambda \) gives us the best underestimator?

\[
\max_\lambda \left( \min_x f(x) + \lambda h(x) \right)
\]
- Lagrangian: \( L(x, \lambda) = f(x) + \lambda h(x) \)
- Dual function: \( g(\lambda) = \min_x L(x, \lambda) = \min_x f(x) + \lambda h(x) \)
Primal problem

\[
\min_x f(x) \quad \text{(P)}
\]

s.t. \( h(x) = 0 \)

Dual problem

\[
\max_\lambda g(\lambda) \quad \text{(D)}
\]

Since (D) is a relaxation of (P), we have

\[ P^* \geq D^* \]
Primal and dual rotation averaging

Primal problem

\[
\begin{align*}
\min_{R} & \quad - \text{tr} \left( R \tilde{R} R^{T} \right) \\
\text{s.t.} & \quad R \in \text{SO}(3)^{n}
\end{align*}
\]  
(P)

Lagrangian

\[
L(R, \Lambda) = - \text{tr} \left( R \tilde{R} R^{T} \right) - \text{tr} \left( \Lambda (I - R^{T} R) \right)
\]

Dual problem

\[
\begin{align*}
\max_{\Lambda - \tilde{R} \succeq 0} & \quad - \text{tr} (\Lambda) \\
\end{align*}
\]  
(D)
Concurrent work

D. Cifuentes, S. Agarwal, P. Parrilo, R. Thomas,
"On the Local Stability of Semidefinite Relaxations", Arxiv 2017
Main Result

Theorem 1 (Strong Duality). Let $R^*_i$, $i = 1, \ldots, n$ denote a stationary point to the primal problem (P) for a connected camera graph $G$ with Laplacian $L_G$. Let $\alpha_{ij}$ denote the angular residuals, i.e., $\alpha_{ij} = \angle(R^*_i, \hat{R}_{ij}, R^*_j)$. Then $R^*_i$, $i = 1, \ldots, n$ will be globally optimal and strong duality will hold for (P) if

$$|\alpha_{ij}| \leq \alpha_{\text{max}} \quad \forall (i, j) \in E,$$

where

$$\alpha_{\text{max}} = 2 \arcsin \left( \sqrt{\frac{1}{4} + \frac{\lambda_2(L_G)}{2d_{\text{max}}} - \frac{1}{2}} \right),$$

and $d_{\text{max}}$ is the maximal vertex degree.

Note: Any local minimizer that fulfills this error bound will be global!
Example: Consider a graph with $n = 3$ vertices that are connected, and all degrees are equal, $d_{max} = 2$. Now from the Laplacian matrix $L_G$, one easily finds that $\lambda_2 = 3$. This gives $\alpha_{max} = \frac{\pi}{3} \text{ rad} = 60^\circ$. So, any local minimizer which has angular residuals less than $60^\circ$ is also a global solution.
Example: For complete graphs,

\[ \alpha_{\text{max}} = 2 \arcsin\left( \frac{\sqrt{3}-1}{2} \right) \approx 0.749 \text{rad} = 42.9^\circ \]
Experiments

| Dataset    | $n$ | Alg. 1 | SeDuMi | $|\alpha_{ij}|$  | $\alpha_{\text{max}}$ |
|------------|-----|--------|--------|----------------|-----------------|
| Gustavus   | 57  | 3.25   | 8.28   | 6.33°          | 8.89°           |
| Sphinx     | 70  | 3.87   | 14.40  | 6.14°          | 12.13°          |
| Alcatraz   | 133 | 12.73  | 117.19 | 7.68°          | 43.15°          |
| Pumpkin    | 209 | 9.23   | 688.65 | 8.63°          | 3.59°           |
| Buddha     | 322 | 16.71  | 1765.72| 7.29°          | 14.01°          |

Table 2: The average run time and largest resulting angular residual ($|\alpha_{ij}|$) and bound ($\alpha_{\text{max}}$) on five different real-world datasets.
Further results

- Full analysis with proofs
- New primal-dual algorithm
- More experimental results

Conclusions

- Strong duality (= zero duality gap) for rotation averaging provided bounded noise levels

- Practically useful sufficient condition for global optimality

- Analysis also leads to efficient algorithm
Future work

- Robust cost functions, e.g., L1 with IRLS
- Further analysis – when is duality gap zero and for what problems?
Point averaging

\[
\min_{\bar{X}_i, R_j, t_j} \sum_{i,j} \| \bar{X}_i - (R_j X_{ij} + t_j) \|^2
\]
Visual localization

High-quality night-time images

Seasonal changes, urban; Low-quality night-time images

Seasonal changes, (sub)urban

www.visuallocalization.net
Benchmark challenge and workshop at CVPR 2019
Estimating a single rotation

\[
\min_R \begin{bmatrix} r_1 \end{bmatrix}^T Q \begin{bmatrix} r_1 \end{bmatrix} \\
\text{where } r = \text{vec}(R), R \in SO(3)
\]

Set \( \Lambda = \begin{bmatrix} r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} r_1 \end{bmatrix}^T \)

\[
\min_\Lambda \text{tr}(Q \Lambda) \\
\Lambda \succeq 0, \text{tr}(A_i \Lambda) = b_i
\]

Convex program, but ignores \( \text{rank}(\Lambda) = 1 \)!

J. Briales and J. Gonzalez-Jimenez, CVPR 2017
Convex Optimization for Rotation Estimation

something else here

A. Eriksson, C. Olsson, F. Kahl, T.-J. Chin
Rotation Averaging and Strong Duality, CVPR 2018