Rotation Averaging and Strong Duality

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CHALMERS

Collaborators

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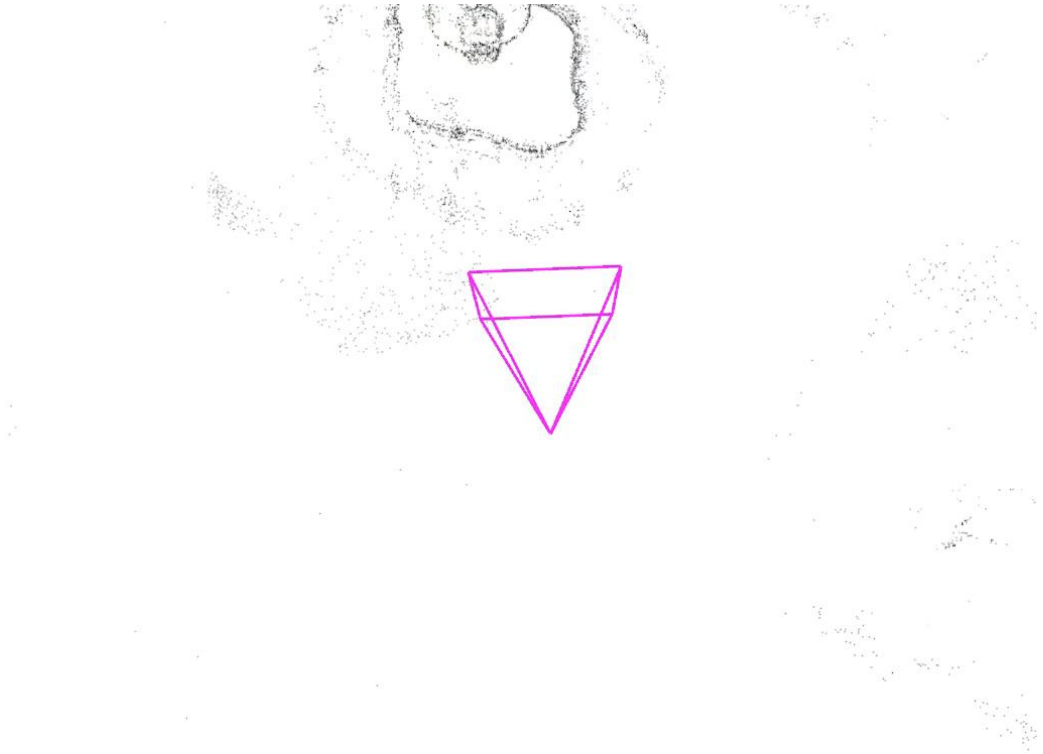
Structure from Motion

Visual Navigation

Visual Localization



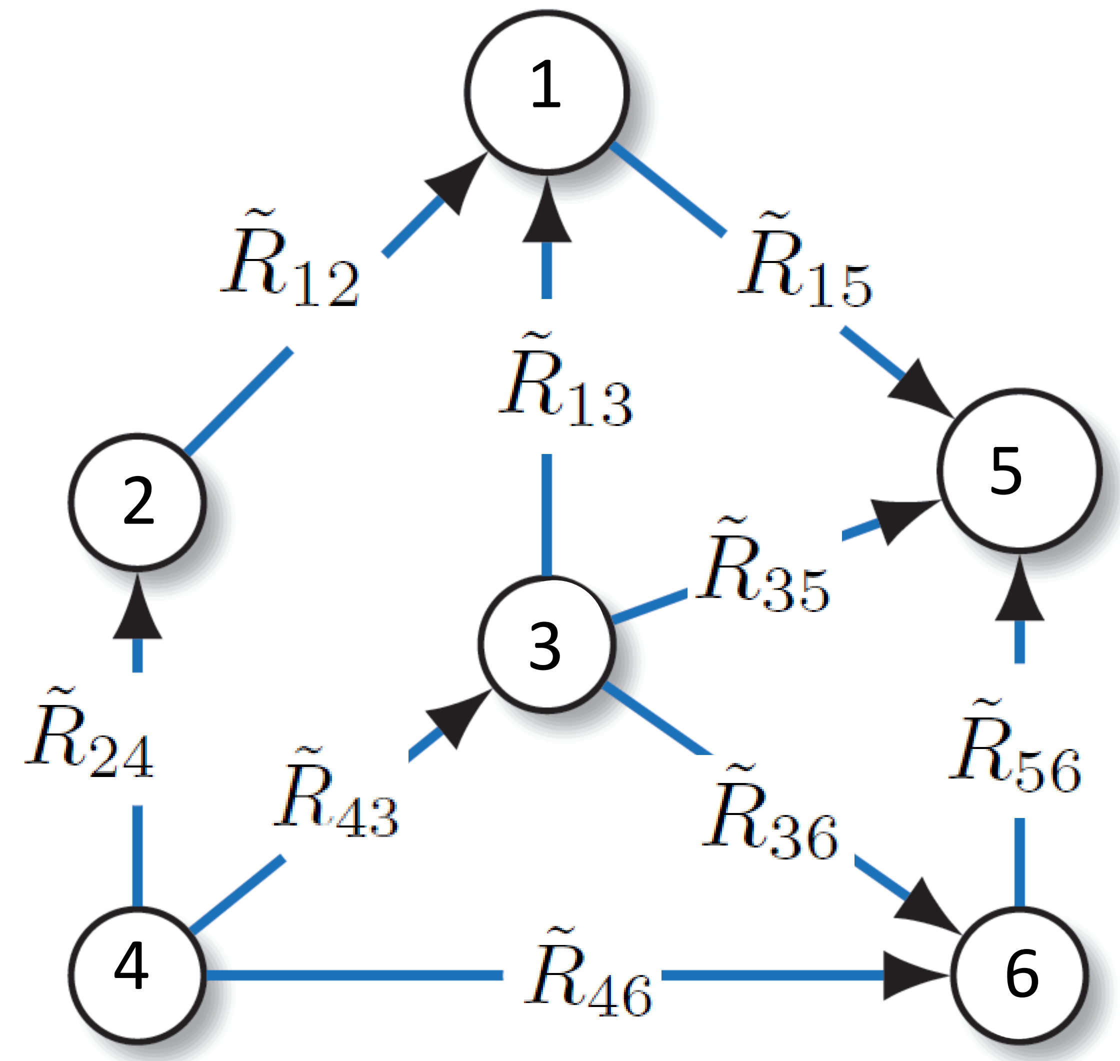
Outline

Main topic: Semidefinite relaxations for optimization over *SO(3)*

- Introduction
- Problem formulation and examples
- Analysis: Relaxations, tightness and extreme points
- In depth: Rotation averaging
- Conclusions

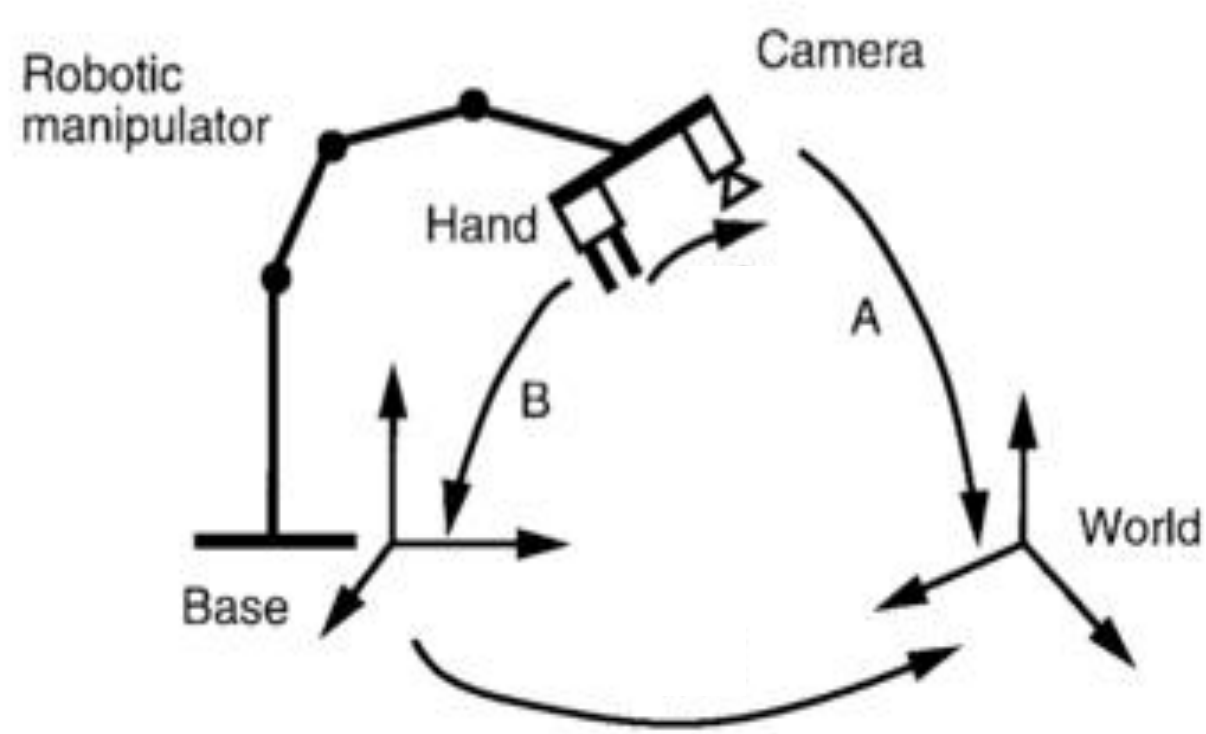
Rotation averaging

- Goal: Recover camera poses given relative pairwise measurements



$$\underset{R_1,\ldots,R_n \in \mathrm{SO}(3)}{\arg\min} \sum_{(i,j)\in E} \|R_i \tilde{R}_{ij} - R_j\|_F^2$$

Hand-eye calibration



$$\min_{R \in \mathrm{SO}(3)} \quad \sum_{i=1}^{m} ||A_i R - R B_i||_F^2$$

The Chordal distance

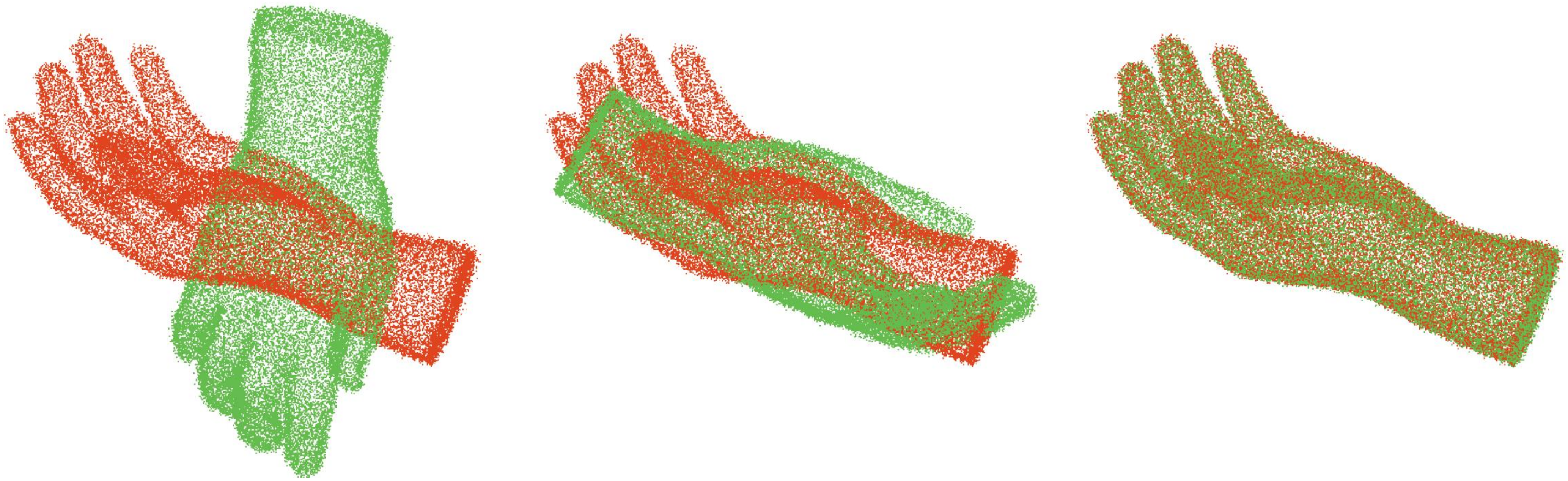
- Defined as the Euclidean distance in the embedding space,

$$d(R, S) = \|R - S\|_F$$

- Equivalent to:

$$d(R, S) = 2\sqrt{2}\sin\frac{|\alpha|}{2}$$

Registration of points, lines and planes



$$\min_{R \in \mathrm{SO}(3),\, t} \quad \sum_{i=1}^{m} \|P_i(Rx_i + t - y_i)\|^2$$

Problem formulation

Let $R = [R_1, \dots R_n],$

where each $R_i \in \mathrm{SO}(3).$

$$\min_{R \in SO(3)^n} \begin{bmatrix} \mathrm{vec}(R) \\ 1 \end{bmatrix}^T Q \begin{bmatrix} \mathrm{vec}(R) \\ 1 \end{bmatrix}$$

How to overcome the problem of non-convexity?

- One idea: Relax some constraints and solve relaxed problem

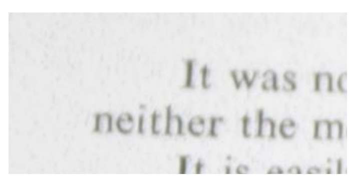
- How to relax?

    1. Linearize
    2. Convexify

- Tightness: When is the solution to the original and relaxed problem the same?

Linearization

- Longuet-Higgins, 1981
- Stefanovic, 1973
- Thompson, 1959
- Chasles, 1855
- Hes
- Hau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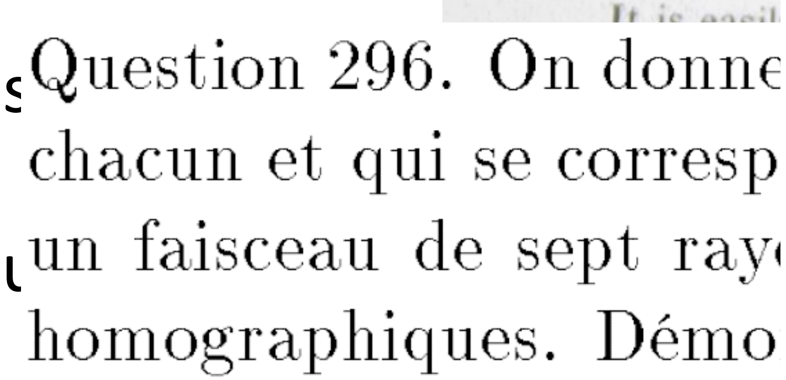
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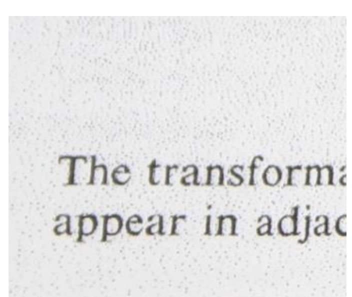
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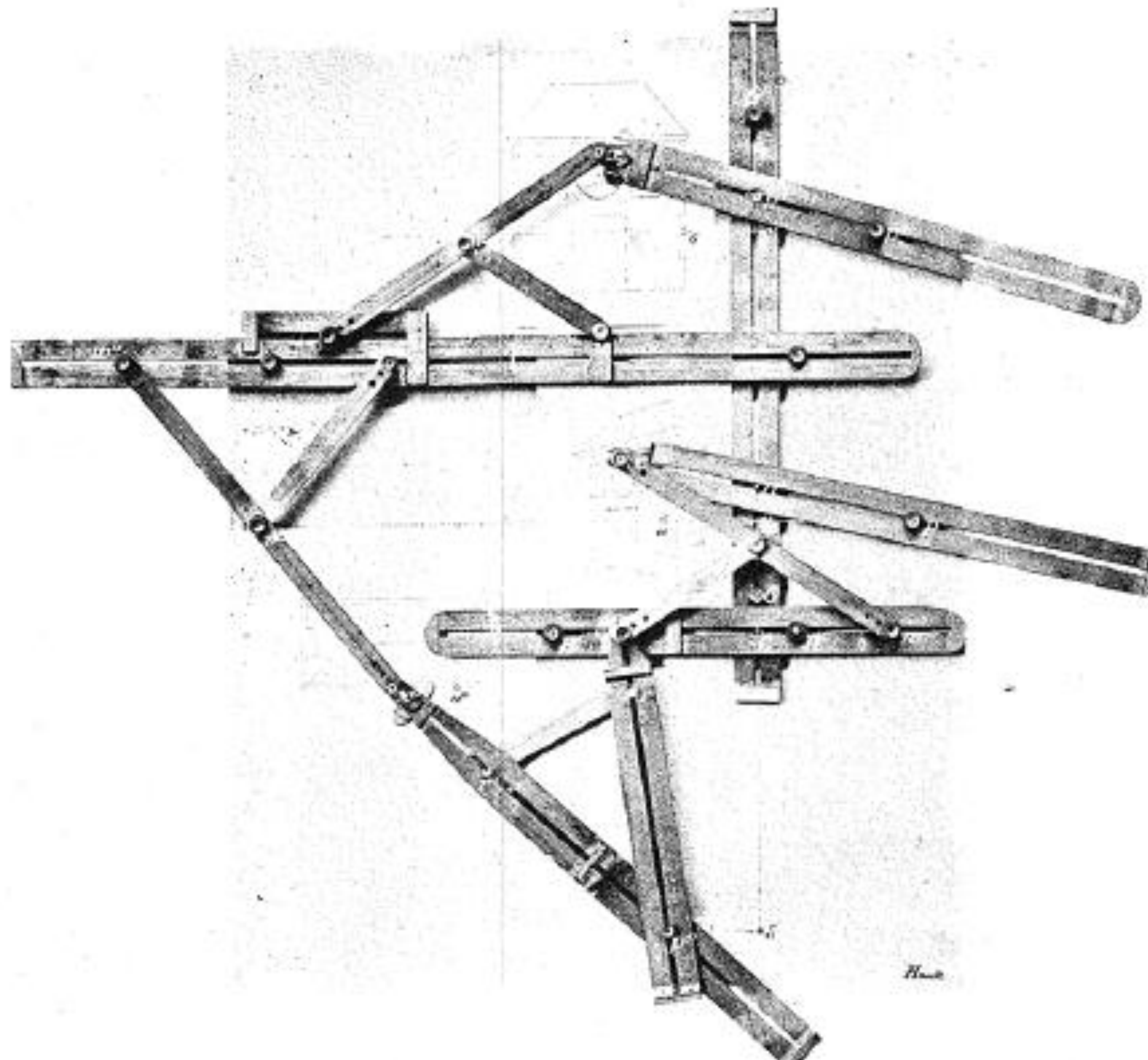


Perspectographe de Guido Hauck

Convexification

- Quasi-convexity

  Q. Ke, T. Kanade, **PAMI 2007**

  F. Kahl, R. Hartley, **PAMI 2008**

- Semidefinite relaxations

  F. Kahl, D. Henrion, **IJCV 2007**

  C. Aholt, S. Agarwal, R. Thomas, **ECCV 2012**

Estimating a single rotation

$$\min_R \quad \begin{bmatrix} r \\ 1 \end{bmatrix}^T Q \begin{bmatrix} r \\ 1 \end{bmatrix}$$
$$r = \mathrm{vec}(R),\, R \in SO(3)$$

Set $\Lambda = \begin{bmatrix} r \\ 1 \end{bmatrix}\begin{bmatrix} r \\ 1 \end{bmatrix}^T$

$$\min_\Lambda \quad \mathrm{tr}(Q\Lambda)$$
$$\Lambda \succeq 0,\ \mathrm{tr}(A_i\Lambda) = b_i$$

Convex program, but ignores $\mathrm{rank}(\Lambda) = 1$!

Estimating a single rotation

Original problem

$$\min_{R} \quad \begin{bmatrix} r \\ 1 \end{bmatrix}^T Q \begin{bmatrix} r \\ 1 \end{bmatrix}$$
$$r = \mathrm{vec}(R), R \in SO(3)$$

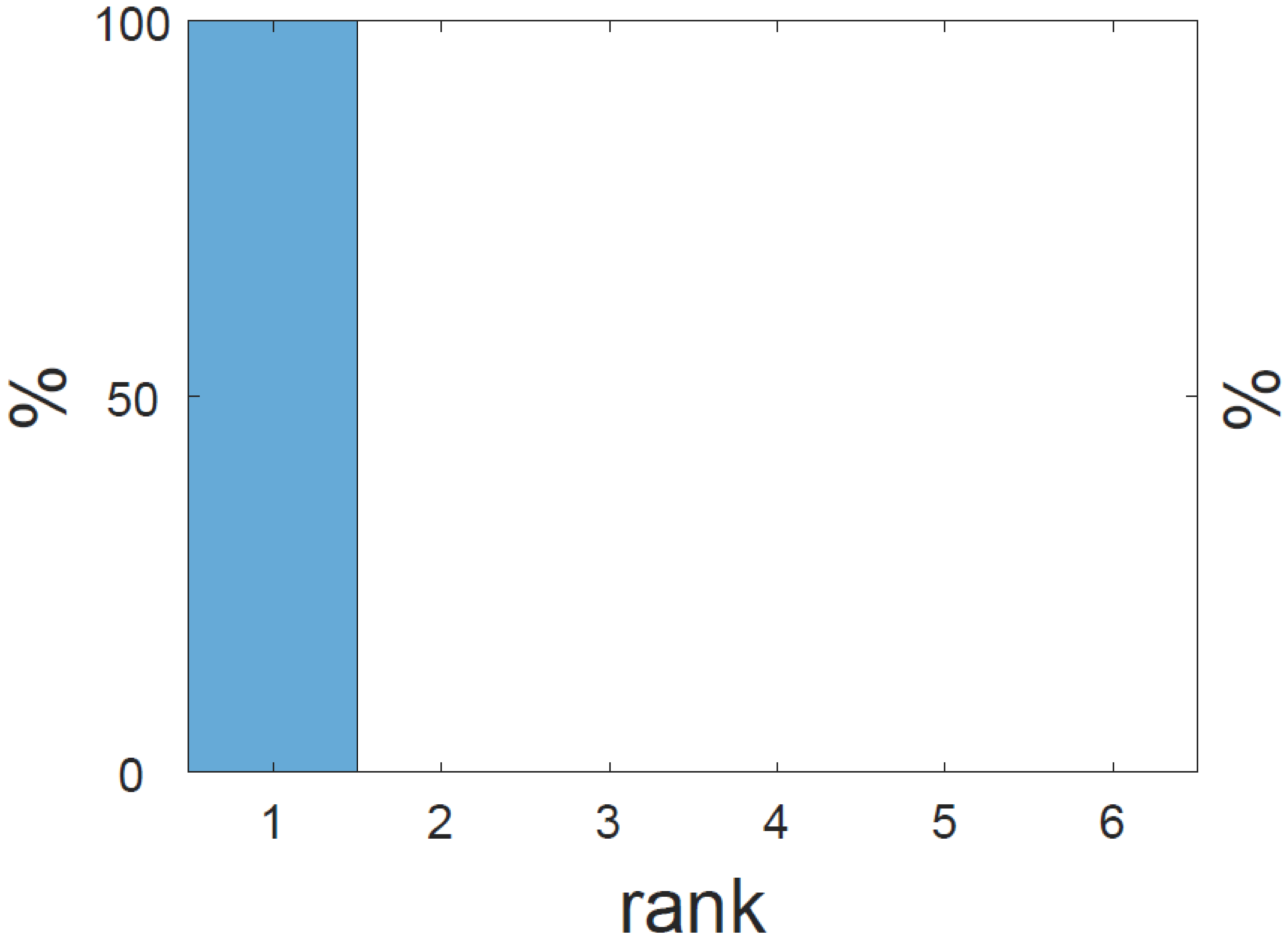
Relaxed problem

$$\min_{\Lambda} \quad \mathrm{tr}(Q\Lambda)$$
$$\Lambda \succeq 0, \; \mathrm{tr}(A_i\Lambda) = b_i$$

- Is the relaxation always tight?

- Are all minimizers Λ^* of the convex relaxation rank one?

Empirical result for 1000 random Q:s



Algebraic Geometry to the rescue

Sums of squares polynomials

Multi-variate polynomial $p(r)$ is a sums of squares (SOS) if

$$p(r) = \sum_i p_i^2(r)$$

Let $R^* \in X$ be a minimizer with optimal value q^*.

Is $\boxed{\begin{bmatrix} r \\ 1 \end{bmatrix}^T Q \begin{bmatrix} r \\ 1 \end{bmatrix} - q^*}$ a sum of squares?

The *SO(3)*-variety

Let X be the variety of 3×3 matrices of $SO(3)$.

$$\mathrm{SO}(3) \in \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I,\ \det(R) = 1\}$$

$$\boxed{\dim(X) = 3, \operatorname{codim}(X) = 6, \operatorname{degree}(X) = 8.}$$

Definition: If non-degenerate variety X has $\operatorname{codim}(X) + 1 = \operatorname{degree}(X)$ then it is called *minimal*.

A theorem by Blekherman *et al*, J. Amer. Math. Soc., 2016

Theorem. *Every real quadratic form that is non-negative on the non-degenerate variety X is a sum of squares of linear forms if and only if X is a variety of minimal degree.*

$$\dim(X) = 3, \operatorname{codim}(X) = 6, \operatorname{degree}(X) = 8.$$

But minimal degree $=$ codim $+ 1 = 7 < 8$.

Extreme points

Original problem

$$\min_{R} \quad \begin{bmatrix} r \\ 1 \end{bmatrix}^T Q \begin{bmatrix} r \\ 1 \end{bmatrix}$$
$$r = \mathrm{vec}(R), R \in SO(3)$$

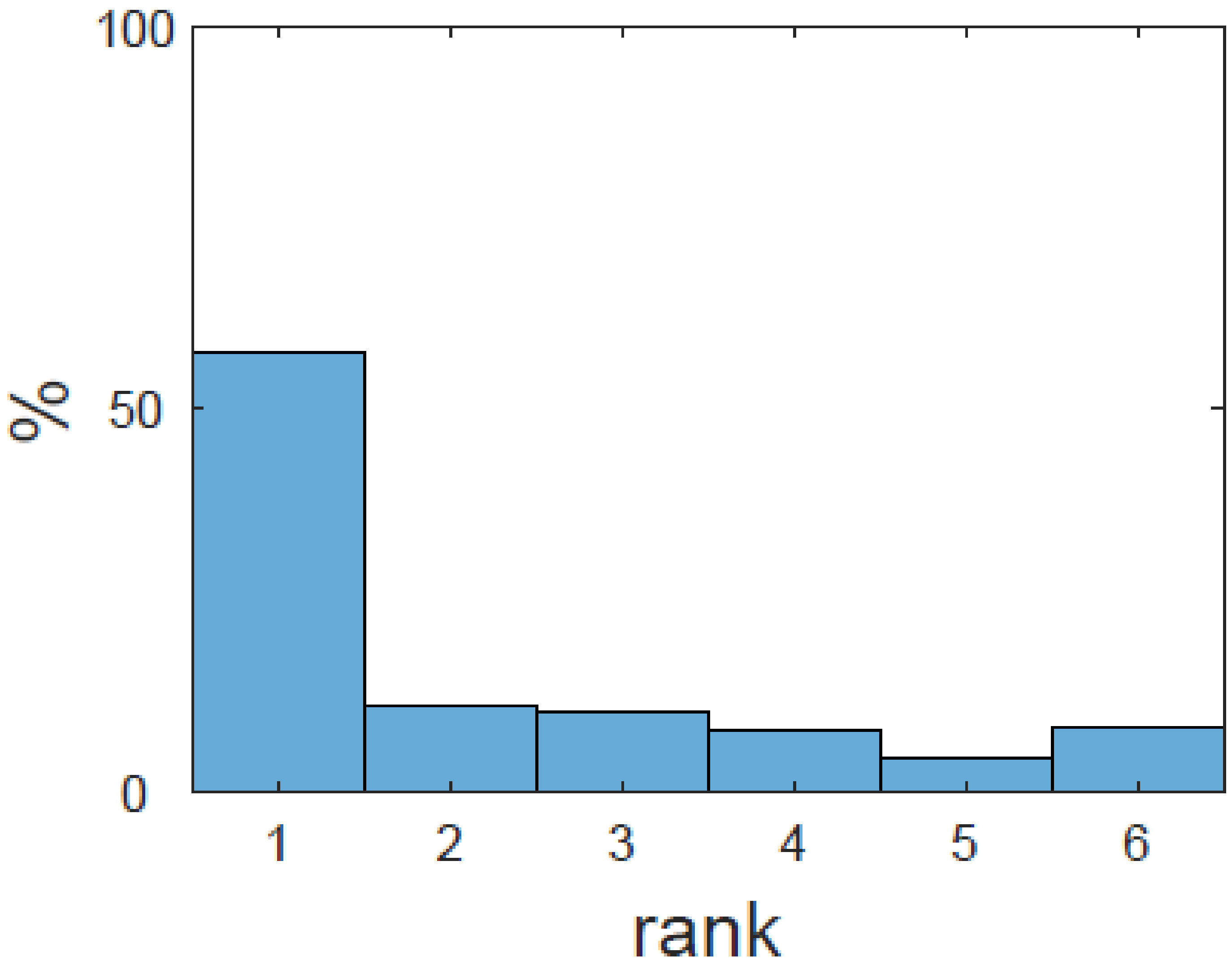
Relaxed problem

$$\min_{\Lambda} \quad \mathrm{tr}(Q\Lambda)$$
$$\Lambda \succeq 0, \ \ \mathrm{tr}(A_i \Lambda) = b_i$$

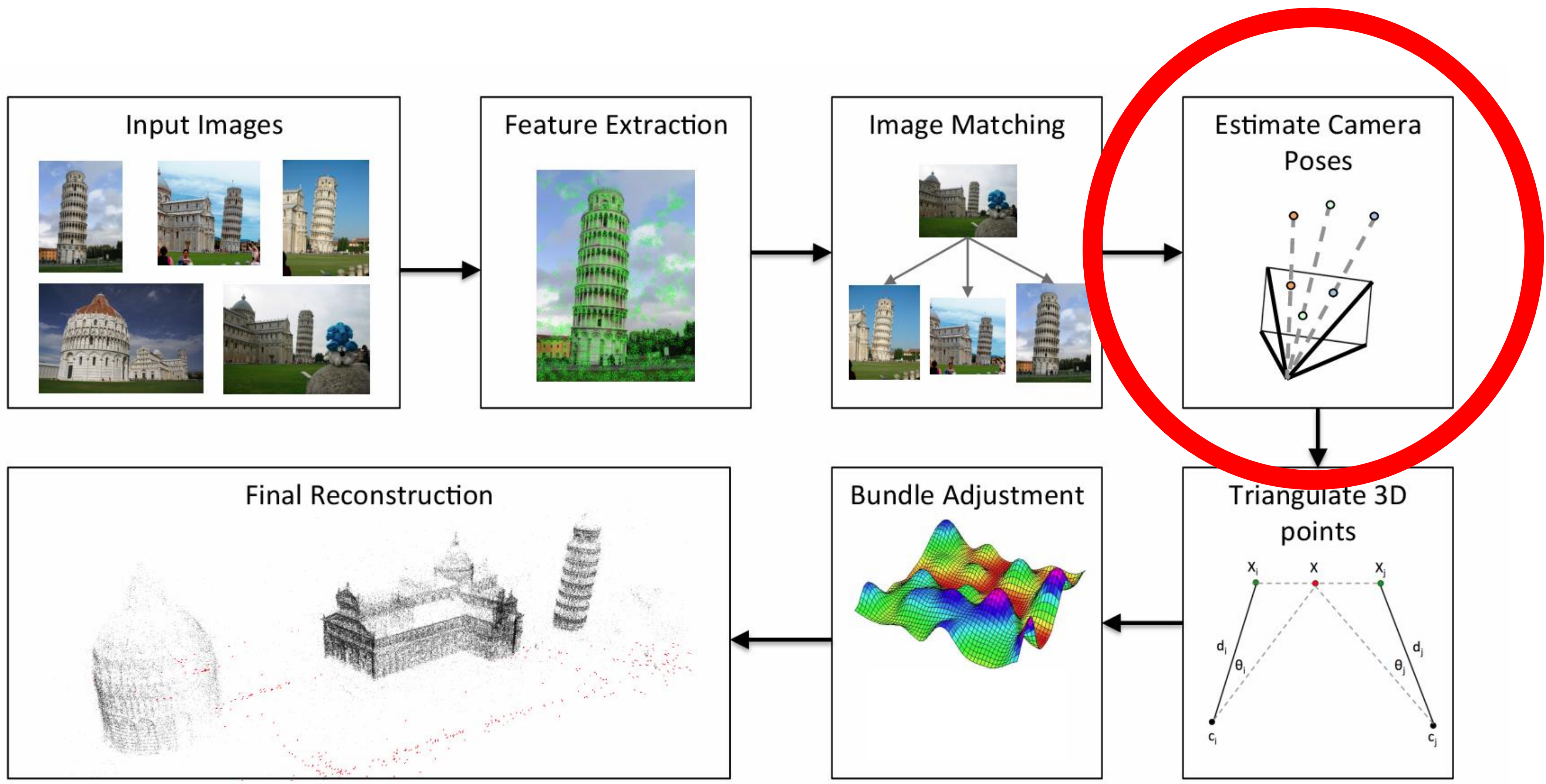
- Are all minimizers Λ^* of the convex relaxation rank one?

Theorem: Every extreme point Λ^* has $\mathrm{rank}(\Lambda^*) = 1$ or $\mathrm{rank}(\Lambda^*) = 6$.

Empirical result for 1000 random Q:s for $SO(3)xSO(3)$



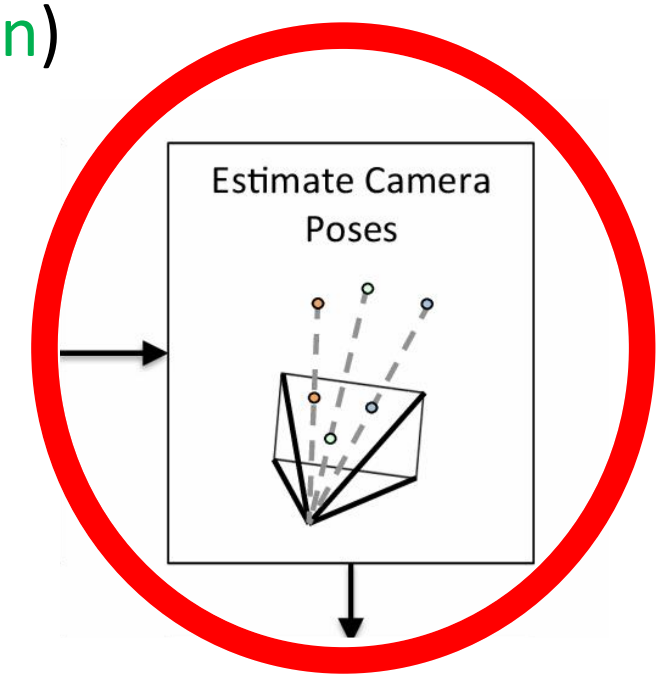
Rotation averaging in Structure from Motion



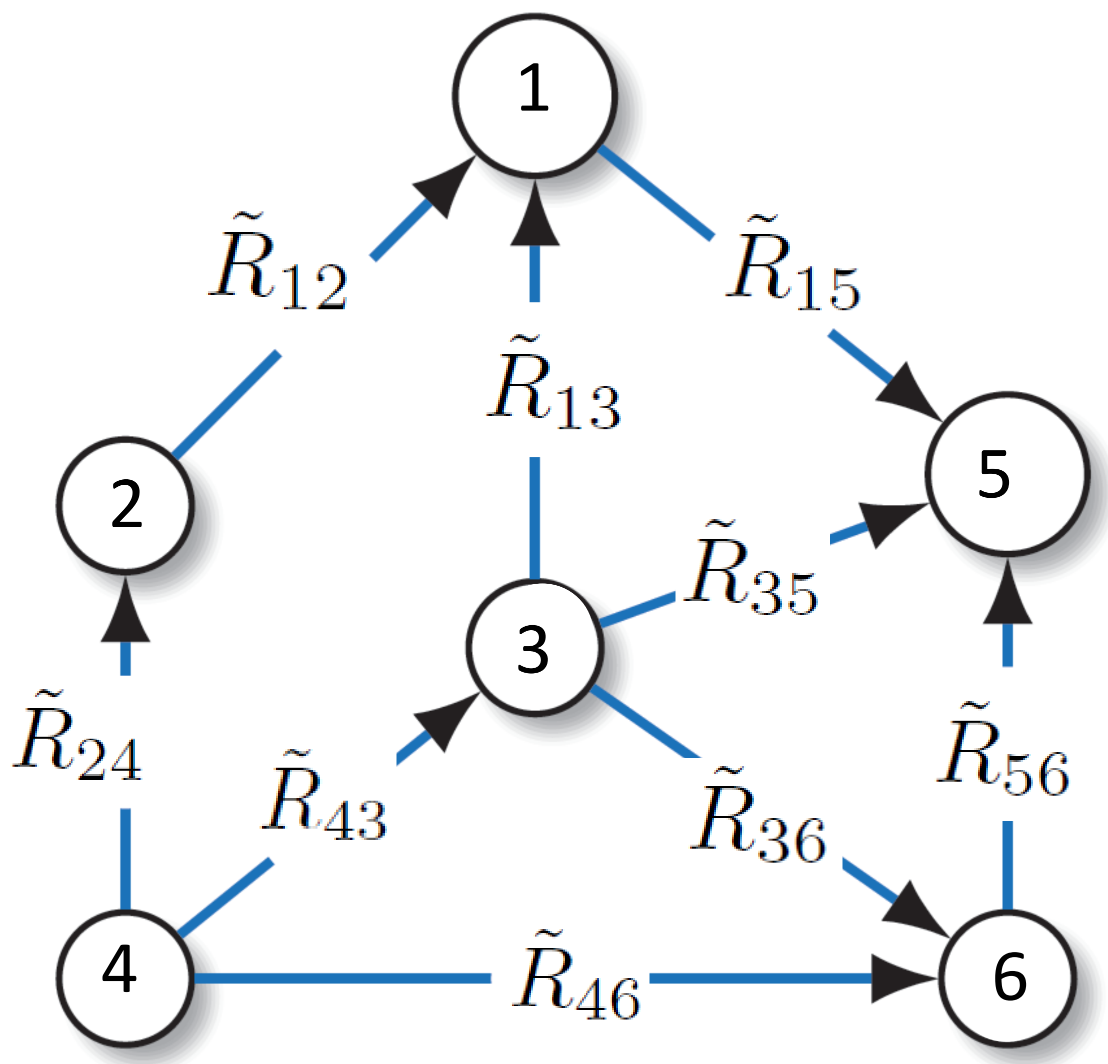
Estimate camera poses

A possible pipeline:

1. Estimate relative epipolar geometries (5-point algorithm)
2. Given relative rotations, estimate absolute rotations
3. Compute camera positions and 3D points (L_∞-optimization)



- Goal: Recover camera poses given relative pairwise measurements



Literature

- Quaternions:

V.M. Govindu, **CVPR 2001**

- Single rotation estimation:

R.I. Hartley, J. Trumpf, Y. Dai and H. Li, **IJCV 2013**

- Duality:

A. Singer, **Applied and Computational Harmonic Analysis, 2011**

J. Fredriksson, C. Olsson, **ACCV 2012**

L. Carlone, D.M. Rosen, G. Calafiore, J.J. Leonard, F. Dellaert, **IROS 2015**

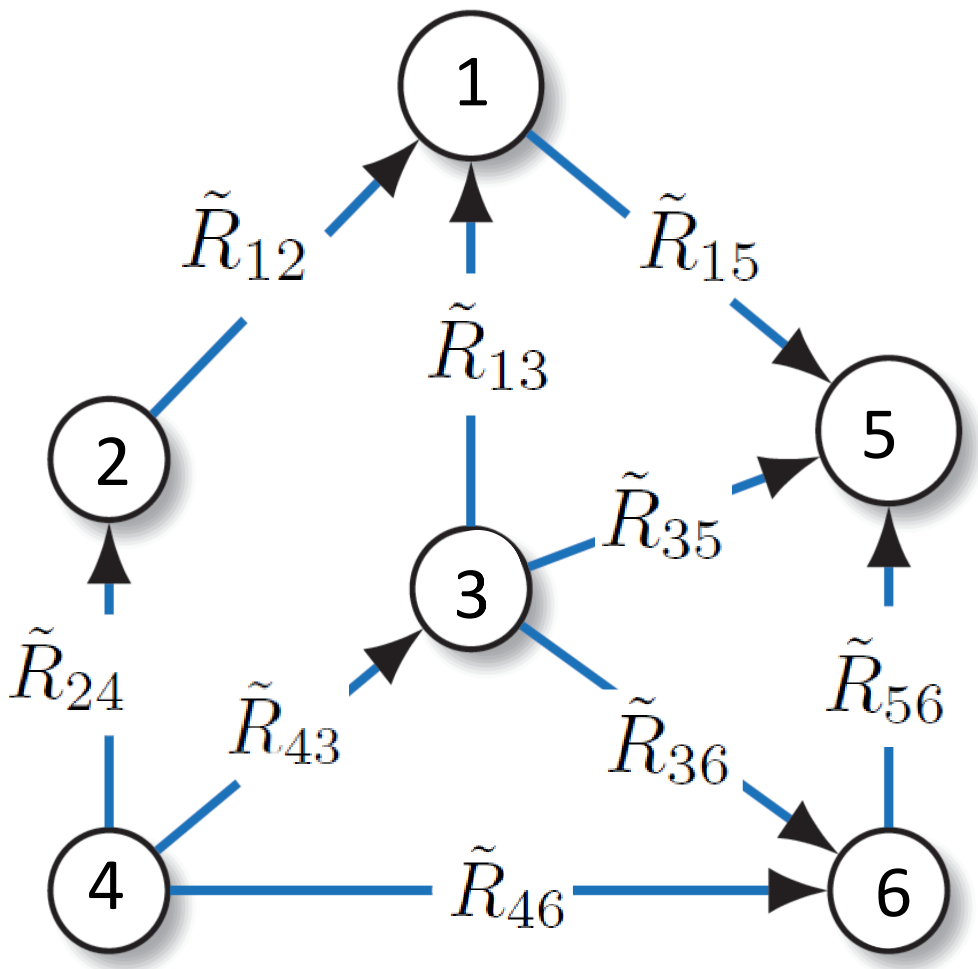
- Analysis:

K. Wilson, D. Bindel and N. Snavely, **ECCV 2016**

- Problem formulation

$$\underset{R_1,\ldots,R_n \in \mathrm{SO}(3)}{\arg\min} \sum_{(i,j)\in E} \|R_i \tilde{R}_{ij} - R_j\|_F^2$$

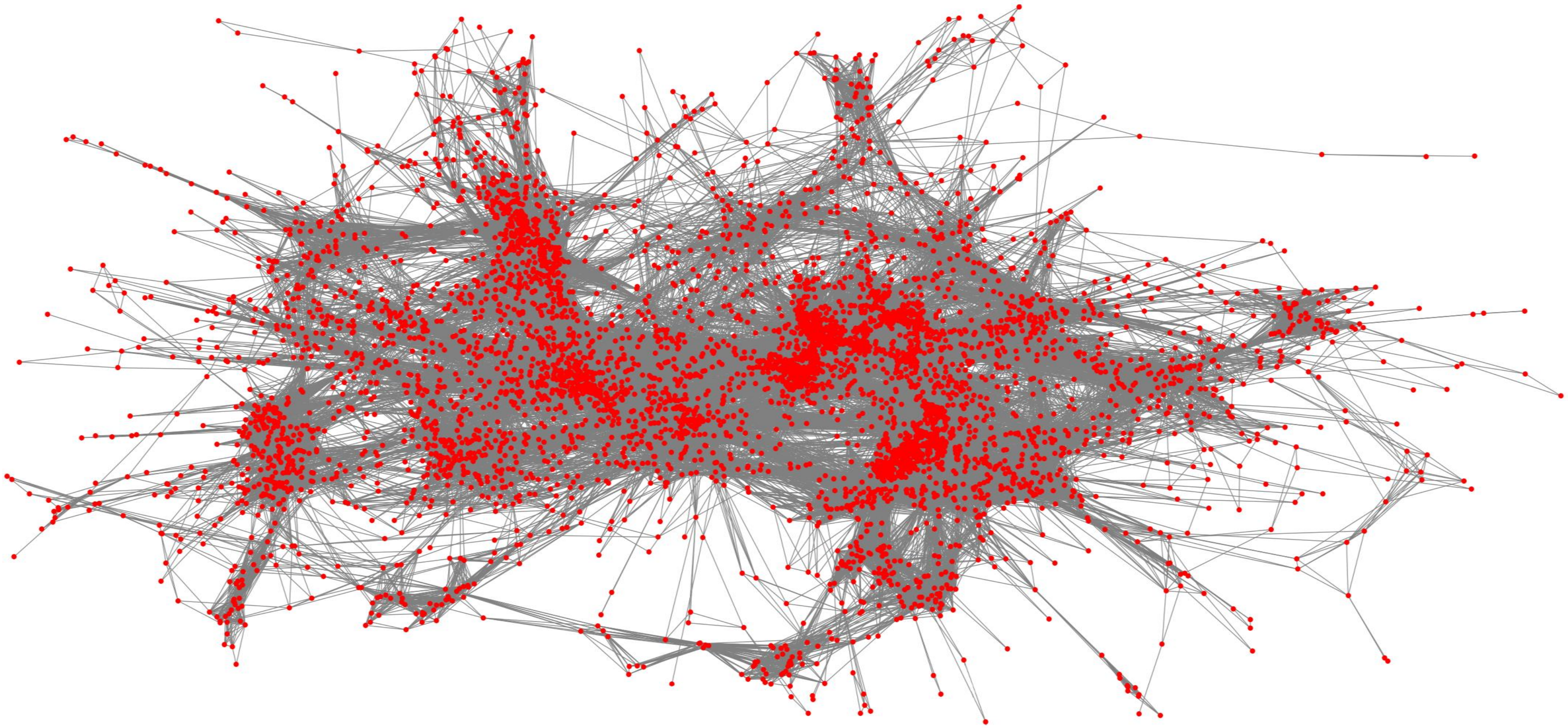
Graph *(V,E)* where *V* = camera poses and *E* = relative rotations



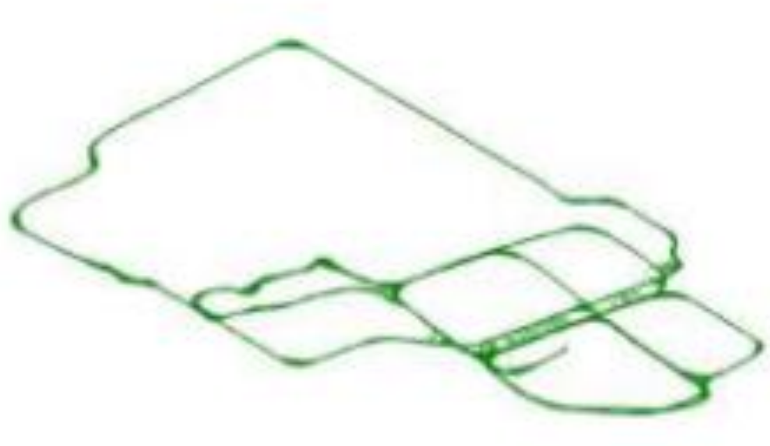
- Problem formulation

$$\arg\min_{R_1,\ldots,R_n \in \mathrm{SO}(3)} \sum_{(i,j)\in E} \|R_i \tilde{R}_{ij} - R_j\|_F^2$$

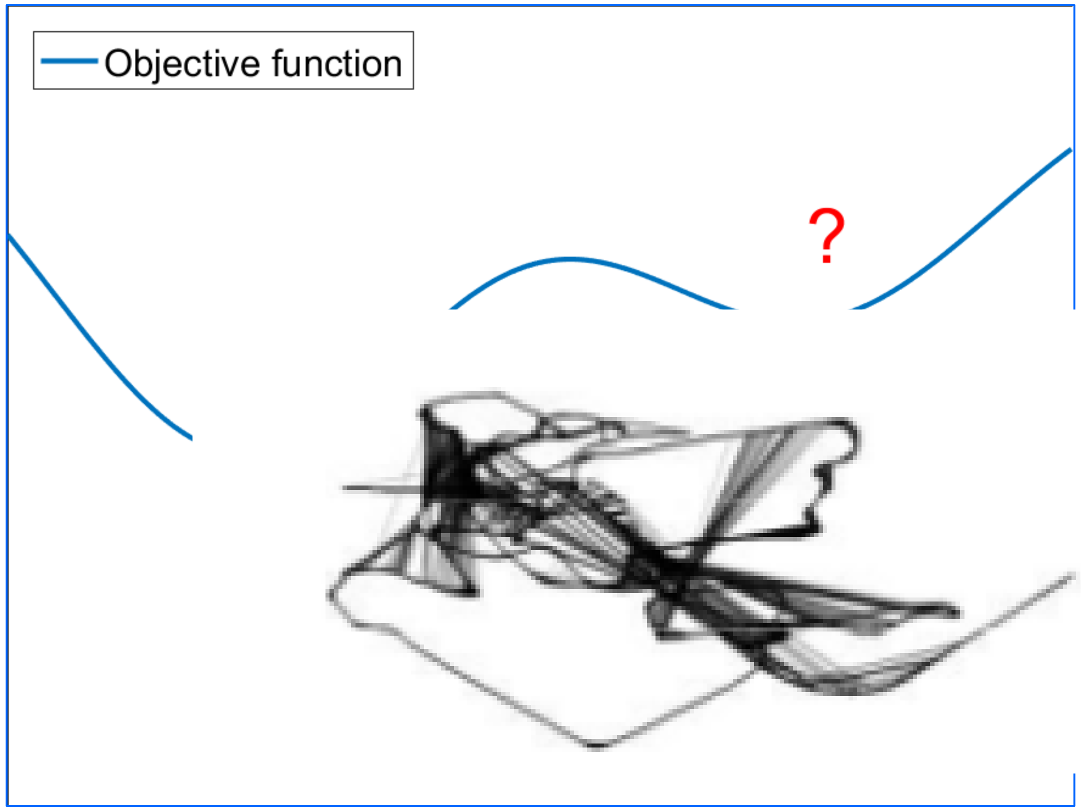
Graph *(V,E)* where *V* = camera poses and *E* = relative rotations



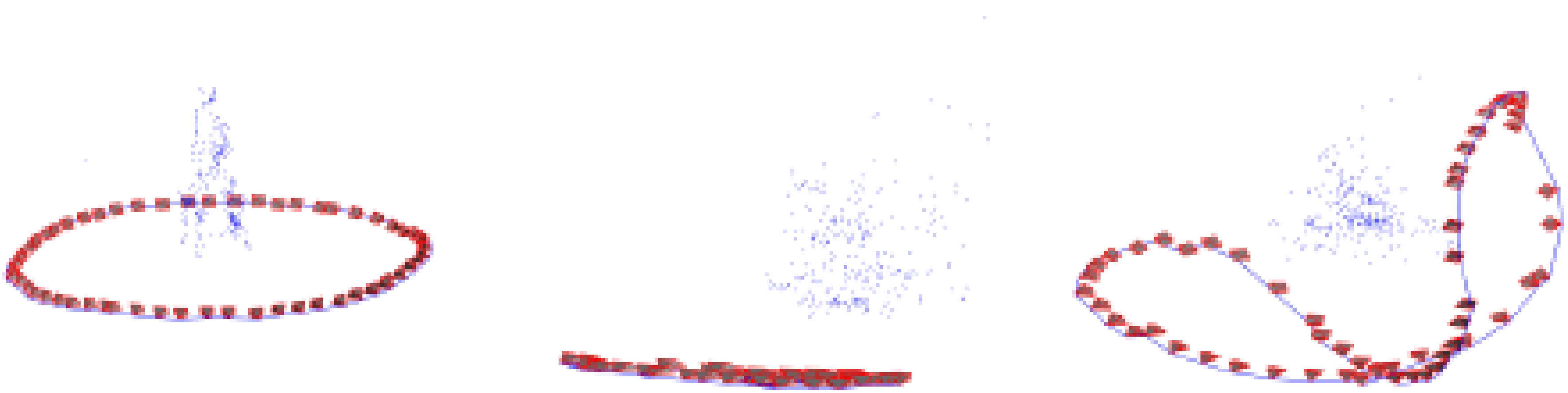
- Non-convex problem



Ground truth



Local minimum



Optimization

- Background
  - Well established theory on duality for convex optimization
  - Duality is at the core of many existing optimization algorithms
  - Less understood about the non-convex case
- Aims
  - Can we obtain guarantees of global optimality?
  - How to design efficient optimization algorithms?

Recall

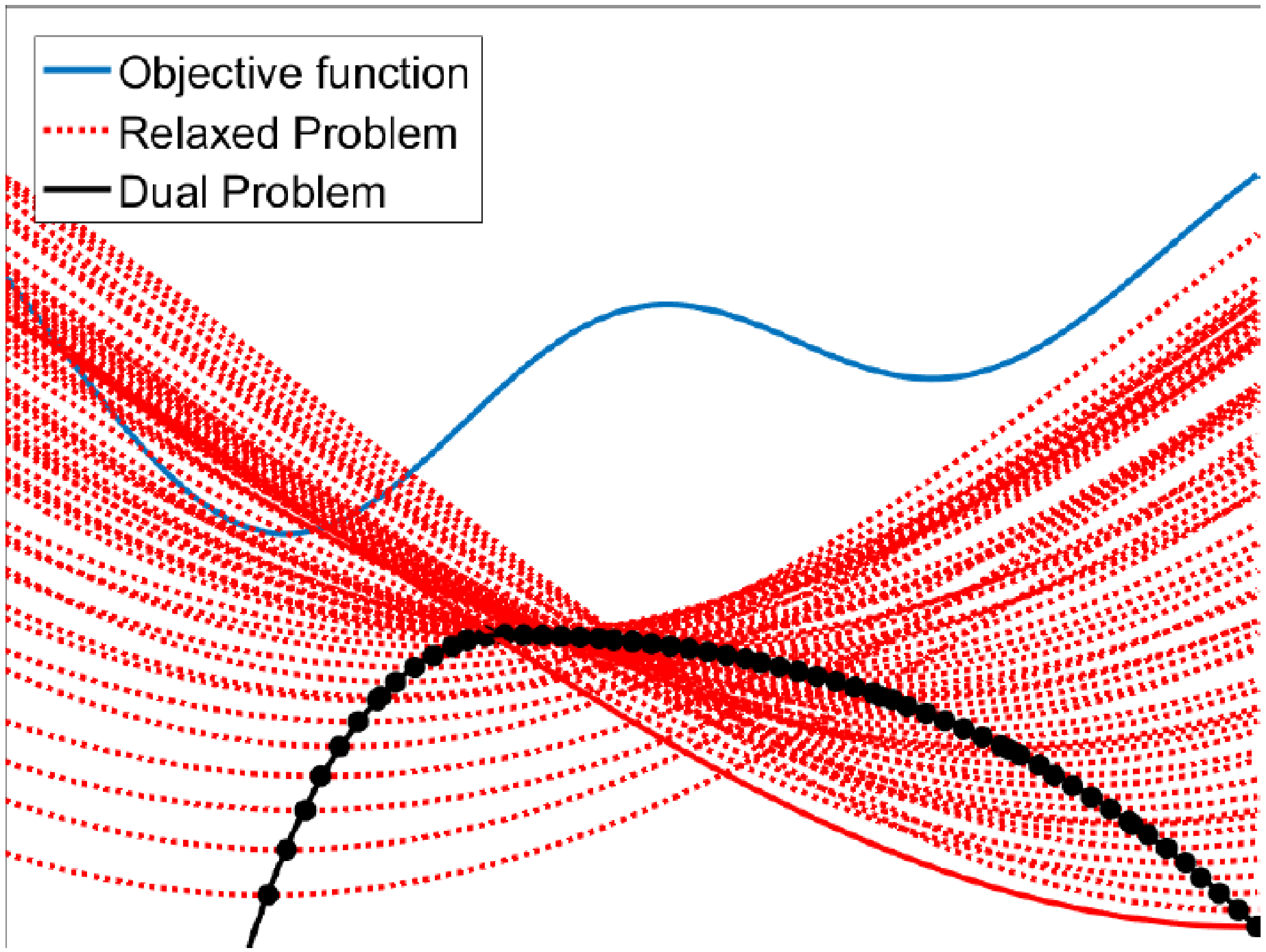
$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & h(x) = 0 \end{aligned} \qquad \geq \qquad \min_{x} \quad f(x) + \lambda h(x)$$

What λ gives us the <u>best</u> underestimator?

$$\max_{\lambda} \left(\min_{x} \quad f(x) + \lambda h(x) \right)$$

Duality

- Lagrangian: $$L(x,\lambda) = f(x) + \lambda h(x)$$
- Dual function: $$g(\lambda) = \min_x L(x,\lambda) = \min_x f(x) + \lambda h(x)$$



Duality

Primal problem

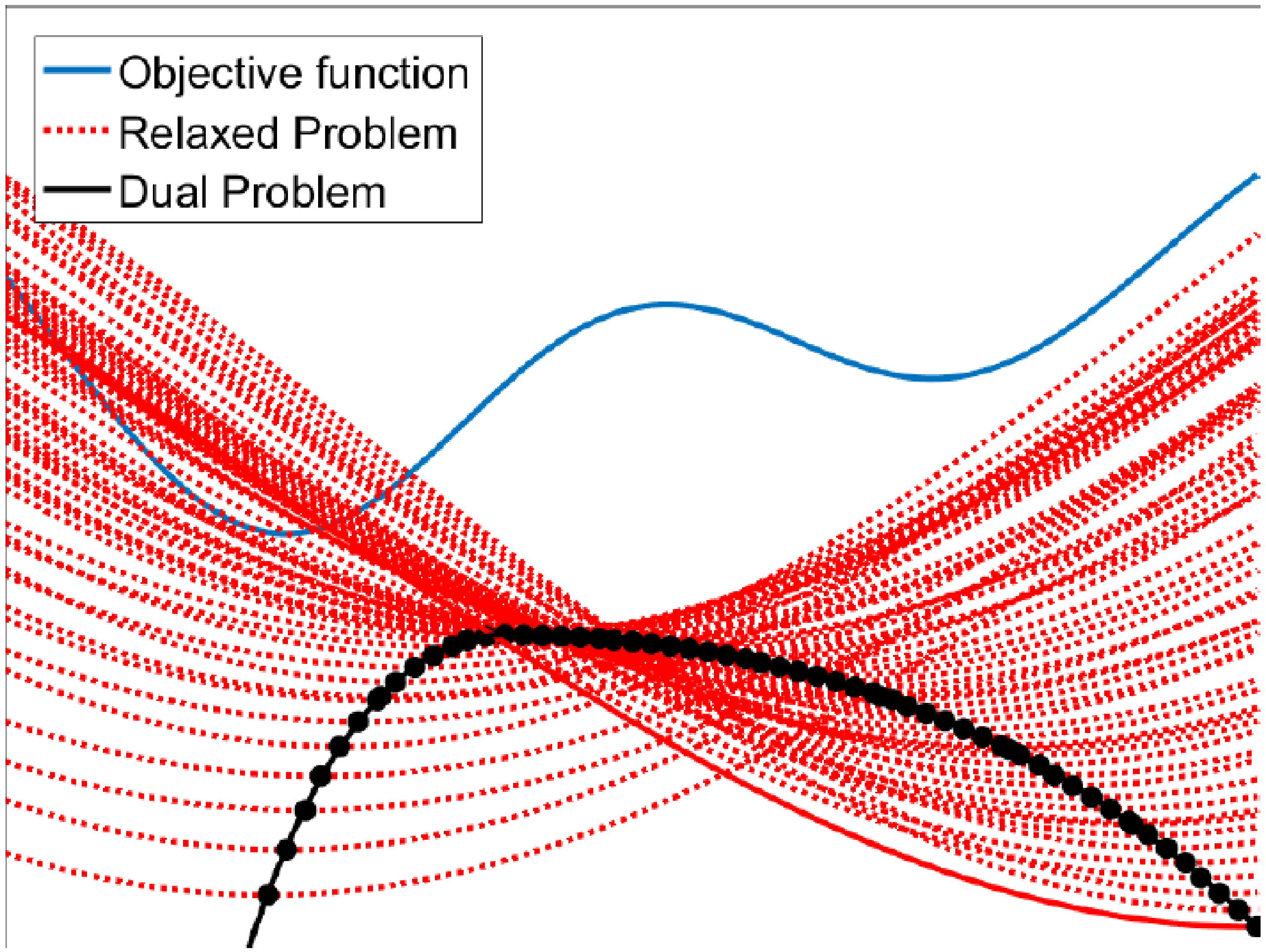
$$\min_{x} \ f(x) \quad \text{s.t.} \ h(x) = 0 \qquad \text{(P)}$$

Dual problem

$$\max_{\lambda} \ g(\lambda) \qquad \text{(D)}$$

Since (D) is a relaxation of (P), we have

$$P^* \geq D^*$$



Primal and dual rotation averaging

Primal problem

$$\begin{aligned} \min \quad & -\mathrm{tr}\left(R\tilde{R}R^T\right) \\ \text{s.t.} \quad & R \in \mathrm{SO}(3)^n \end{aligned} \qquad \text{(P)}$$

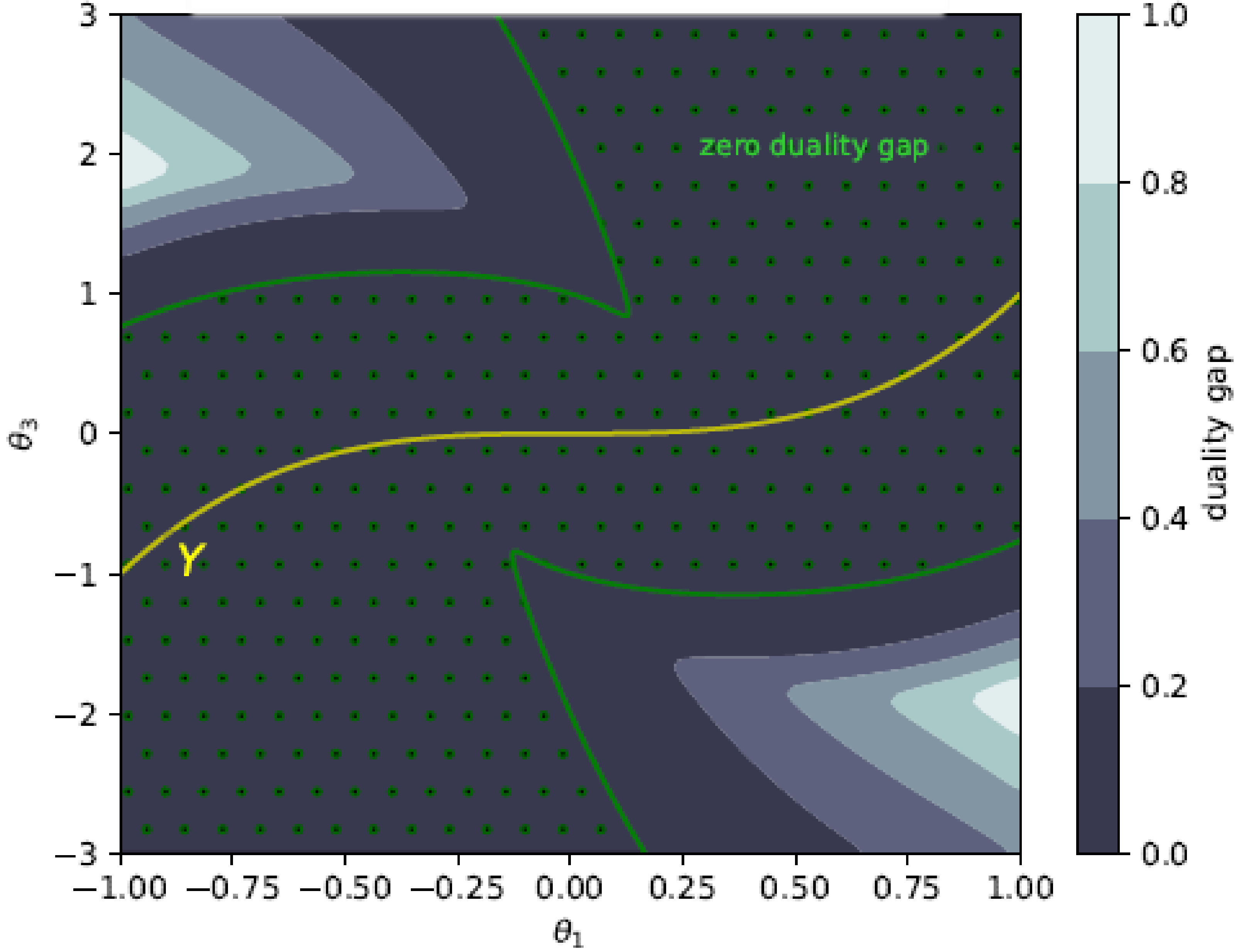
Lagrangian

$$L(R,\Lambda) \quad = -\mathrm{tr}\left(R\tilde{R}R^T\right) - \mathrm{tr}\left(\Lambda(I - R^T R)\right)$$

Dual problem

$$\max_{\Lambda - \tilde{R} \succeq 0} -\mathrm{tr}\left(\Lambda\right) \qquad \text{(D)}$$

Concurrent work



D. Cifuentes, S. Agarwal, P. Parrilo, R. Thomas,
"On the Local Stability of Semidefinite Relaxations", **Arxiv 2017**

Main Result

Theorem 1 (Strong Duality). *Let R_i^*, $i = 1, \dots, n$ denote a stationary point to the primal problem (P) for a connected camera graph G with Laplacian L_G. Let α_{ij} denote the angular residuals, i.e., $\alpha_{ij} = \angle(R_i^* \tilde{R}_{ij}, R_j^*)$. Then R_i^*, $i = 1, \dots, n$ will be globally optimal and strong duality will hold for (P) if*

$$|\alpha_{ij}| \leq \alpha_{\max} \quad \forall (i,j) \in E, \tag{1}$$

where

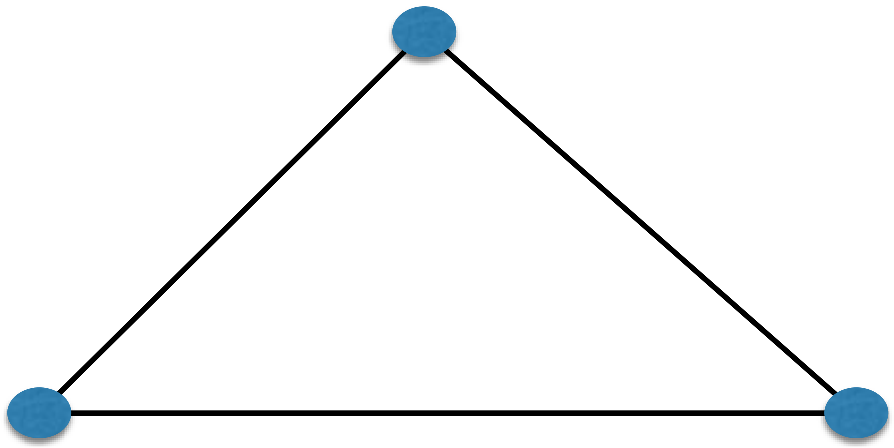
$$\alpha_{\max} = 2 \arcsin \left(\sqrt{\frac{1}{4} + \frac{\lambda_2(L_G)}{2 d_{\max}}} - \frac{1}{2} \right), \tag{2}$$

and $d_{\max}$ is the maximal vertex degree.

Note : Any local minimizer that fulfills this error bound will be global!

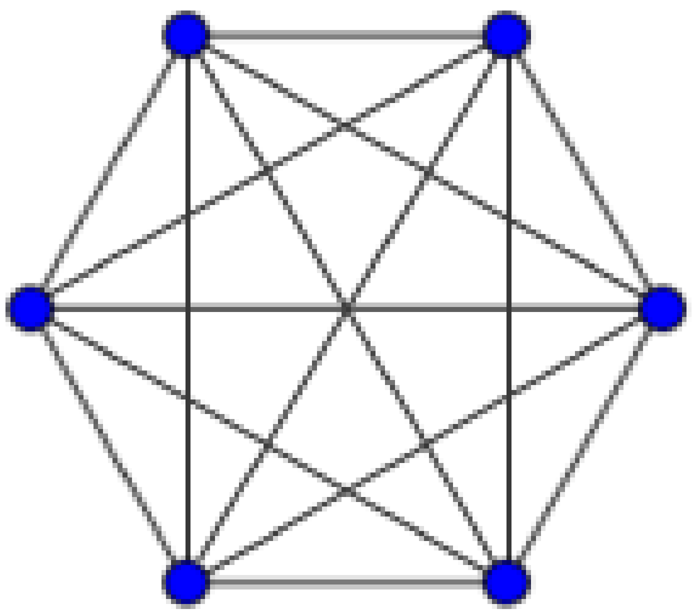
Corollaries

Example: Consider a graph with $n = 3$ vertices that are connected, and all degrees are equal, $d_{max} = 2$. Now from the Laplacian matrix L_G, one easily finds that $\lambda_2 = 3$. This gives $\alpha_{\max} = \frac{\pi}{3}\text{rad} = 60°$. So, any local minimizer which has angular residuals less than $60°$ is also a global solution.



Example: For complete graphs,

$$\alpha_{\max} = 2\arcsin\left(\frac{\sqrt{3}-1}{2}\right) \approx 0.749\mathrm{rad} = 42.9^{\circ}$$



Experiments

		$time[s]$			
Dataset	n	Alg. 1	SeDuMi	$\lvert\alpha_{ij}\rvert$	$\alpha_{\max}$
Gustavus	57	3.25	8.28	$6.33°$	$8.89°$
Sphinx	70	3.87	14.40	$6.14°$	$12.13°$
Alcatraz	133	12.73	117.19	$7.68°$	$43.15°$
Pumpkin	209	9.23	688.65	$8.63°$	$3.59°$
Buddha	322	16.71	1765.72	$7.29°$	$14.01°$

Table 2: The average run time and largest resulting angular residual ($\lvert\alpha_{ij}\rvert$) and bound ($\alpha_{\max}$) on five different real-world datasets.

Further results

- Full analysis with proofs
- New primal-dual algorithm
- More experimental results

A. Eriksson, C. Olsson, F. Kahl, T.J. Chin, **to appear PAMI 2019**

Conclusions

- Strong duality (= zero duality gap) for rotatation averaging provided bounded noise levels

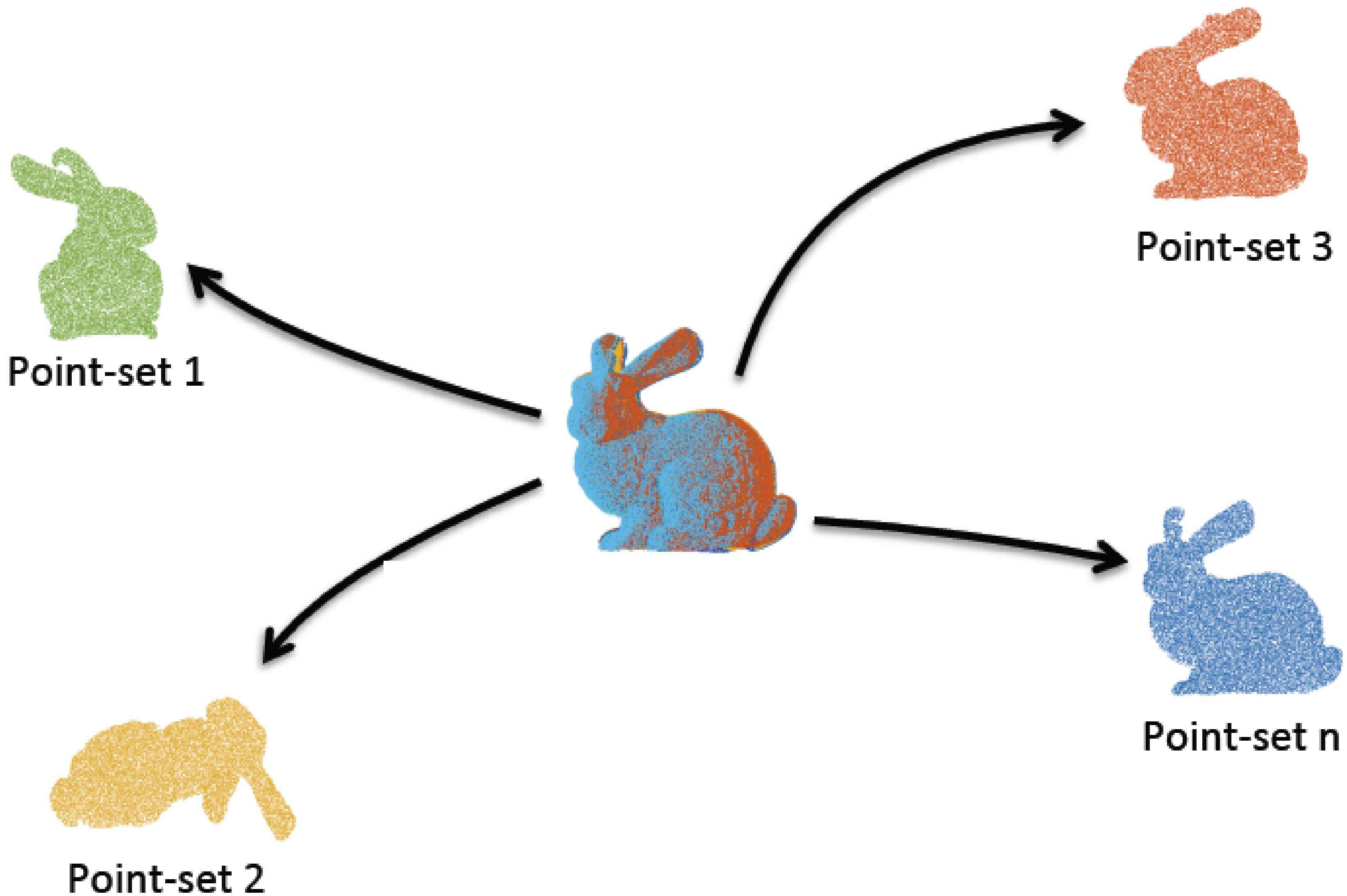
- Practically useful sufficient condition for global optimality

- Analysis also leads to efficient algorithm

Future work

- Robust cost functions, e.g., L1 with IRLS
- Further analysis – when is duality gap zero and for what problems?

Point averaging



$$\min_{\bar{X}_i, R_j, t_j} \sum_{i,j} \|\bar{X}_i - (R_j X_{ij} + t_j)\|^2$$

Visual localization

High-quality night-time images

Seasonal changes, (sub)urban

Seasonal changes, urban; Low-quality night-time images

www.visuallocalization.net

Benchmark challenge and workshop at CVPR 2019

Estimating a single rotation

$$\min_{R} \quad \begin{bmatrix} r \\ 1 \end{bmatrix}^T Q \begin{bmatrix} r \\ 1 \end{bmatrix}$$
$$r = \text{vec}(R), R \in SO(3)$$

Set $\Lambda = \begin{bmatrix} r \\ 1 \end{bmatrix}\begin{bmatrix} r \\ 1 \end{bmatrix}^T$

$$\min_{\Lambda} \quad \text{tr}(Q\Lambda)$$
$$\Lambda \succeq 0, \;\; \text{tr}(A_i \Lambda) = b_i$$

Convex program, but ignores $\text{rank}(\Lambda) = 1$!

J. Briales and J. Gonzalez-Jimenez, **CVPR 2017**

Convex Optimization for Rotation Estimation

something else here

A. Eriksson, C. Olsson, F. Kahl, T.-J. Chin
Rotation Averaging and Strong Duality, **CVPR 2018**